

# Cluster Algebras and Symmetries of Regular Tilings

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## Abstract

The classification of Grassmannian cluster algebras resembles that of regular polygonal tilings. We conjecture that this resemblance may indicate a deeper connection between these seemingly unrelated structures.

## 1 Introduction

Grassmannian cluster algebras and regular tilings (and the symmetry groups of those tilings) obey similar classifications into finite, affine, and hyperbolic (or spherical, planar, and hyperbolic) cases. In fact, one can construct identical tables of cluster algebras and tilings, as shown below.

Formally, we observe that **the cluster algebra  $\text{Gr}(p, p+q)$  is finite iff the Coxeter group  $[p, q]$  is finite, extended-affine iff  $[p, q]$  is affine, and hyperbolic iff  $[p, q]$  is hyperbolic.**

Or, in simpler language:  **$\text{Gr}(p, p+q)$  is of finite type iff the regular tiling  $\{p, q\}$  is spherical, infinite but of finite mutation type iff  $\{p, q\}$  is planar, and of infinite mutation type iff  $\{p, q\}$  is hyperbolic.**

We conjecture that this similarity follows from a deeper connection between these structures. Understanding such a connection could provide a shorter, more elegant classification theorem for Grassmannian cluster algebras.

## 2 Grassmannian Cluster Algebras

Scott [2003] and Fomin et al. [2007] have proven that the following classification of Grassmannian cluster algebras  $\text{Gr}(p, p+q)$  (with subspace dimension  $p$  and codimension  $q$ ) holds:

$p \backslash q$	2	3	4	5	6	7
2	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$
3	$A_2$	$D_4$	$E_6$	$E_8$	$E_8^{(1,1)}$	$\text{Gr}(3,10)$
4	$A_3$	$E_6$	$E_7^{(1,1)}$	$\text{Gr}(4,9)$	$\text{Gr}(4,10)$	$\text{Gr}(4,11)$
5	$A_4$	$E_8$	$\text{Gr}(5,9)$	$\text{Gr}(5,10)$	$\text{Gr}(5,11)$	$\text{Gr}(5,12)$
6	$A_5$	$E_8^{(1,1)}$	$\text{Gr}(6,10)$	$\text{Gr}(6,11)$	$\text{Gr}(6,12)$	$\text{Gr}(6,13)$
7	$A_6$	$\text{Gr}(7,10)$	$\text{Gr}(7,11)$	$\text{Gr}(7,12)$	$\text{Gr}(7,13)$	$\text{Gr}(7,14)$

- Green cells of the table denote finite cluster algebras, with finite Dynkin diagrams  $X_n$ .
- Yellow cells are infinite, but of finite mutation type, with extended affine Dynkin diagrams  $X_n^{(1,1)}$ .
- Red cells are of infinite mutation type, with hyperbolic Dynkin diagrams.

This classification depends on the parameter  $r = (p-2)(q-2)$  (see Fomin et al. [2007] Prop. 12.11): **Gr( $p, p+q$ ) is finite for  $r < 4$ , infinite with finite mutation type for  $r = 4$ , and infinite mutation type for  $r > 4$ .**

Fomin et al. [2007] prove this statement by considering individual cases.

There is a canonical isomorphism  $\text{Gr}(p, p+q) \cong \text{Gr}(q, q+p)$ .

### 3 Regular Tilings

This table depicts regular tilings  $\{p, q\}$ . These are two-dimensional surfaces formed by joining together regular  $p$ -gons, with  $q$  at each vertex.

$p \setminus q$	2	3	4	5	6	7
2	$\{2,2\}$	$\{2,3\}$	$\{2,4\}$	$\{2,5\}$	$\{2,6\}$	$\{2,7\}$
3	$\{3,2\}$	tetrahedron	octahedron	icosahedron	triangular tiling	$\{3,7\}$
4	$\{4,2\}$	cube	square tiling	$\{4,5\}$	$\{4,6\}$	$\{4,7\}$
5	$\{5,2\}$	dodecahedron	$\{5,4\}$	$\{5,5\}$	$\{5,6\}$	$\{5,7\}$
6	$\{6,2\}$	hexagonal tiling	$\{6,4\}$	$\{6,5\}$	$\{6,6\}$	$\{6,7\}$
7	$\{7,2\}$	$\{7,3\}$	$\{7,4\}$	$\{7,5\}$	$\{7,6\}$	$\{7,7\}$

- Green cells are spherical tilings: hosohedra  $\{2, q\}$ , dihedra  $\{p, 2\}$ , and the five Platonic solids.
- The three yellow cells are planar tilings.
- The red cells are regular hyperbolic tilings.

The nature of such a tiling depends on the value of  $r = (p-2)(q-2)$ : it will be spherical for  $r < 4$ , planar for  $r = 4$ , and hyperbolic for  $r > 4$ . This can be shown by calculating the angular defect at each vertex.

The tilings  $\{p, q\}$  and  $\{q, p\}$  are dual.

#### 3.1 Symmetry Groups of Tilings

The tiling  $\{p, q\}$  has the symmetry group  $[p, q]$  in Coxeter notation; these have associated Coxeter-Dynkin diagrams and obey a Cartan-Killing classification, much like cluster algebras.

- Spherical tilings correspond to finite Coxeter groups  $X_n$ .
- Planar tilings have affine Coxeter groups  $X_n^{(1)}$  (also called  $\tilde{X}_n$ ).
- Hyperbolic tilings have hyperbolic Coxeter groups.

Dual tilings have the same symmetries,  $[p, q] \cong [q, p]$ .

## 4 Dynkin diagrams

Note that  $[p, q]$  and  $\text{Gr}(p, p + q)$  have very different, and apparently unrelated, Dynkin diagrams. In particular, the diagram for  $[p, q]$  is of rank 3 while  $\text{Gr}(p, p + q)$  has rank  $(p - 1)(q - 1)$ . The initial quiver of  $\text{Gr}(p, p + q)$  is mutation-equivalent to its Dynkin diagram, which is also of this rank.

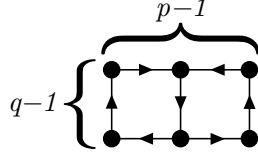


Figure 1: The initial quiver of  $\text{Gr}(p, p + q)$



Figure 2: The Coxeter-Dynkin diagram of  $[p, q]$

## 5 Summary Table

$r$	Cluster Algebra		Regular Tiling	Symmetries	Coxeter Notation
$r = 0$	<b>simplest finite type</b> $\text{Gr}(2, p + 2) \cong \text{Gr}(p, p + 2)$	$A_{p-1}$	<b>degenerate spherical</b> hosohedron $\cong$ dihedron	$A_1 \times I_2(p)$	$[2, p] \cong [p, 2]$
$0 < r < 4$	<b>other finite type</b> $\text{Gr}(3, 6)$ $\text{Gr}(3, 7) \cong \text{Gr}(4, 7)$ $\text{Gr}(3, 8) \cong \text{Gr}(5, 8)$	$D_4$ $E_6$ $E_8$	<b>Platonic</b> tetrahedron octahedron $\cong$ cube icosahedron $\cong$ dodecahedron	$A_3$ $BC_3$ $H_3$	$[3, 3]$ $[3, 4] \cong [4, 3]$ $[3, 5] \cong [5, 3]$
$r = 4$	<b>finite mutation type</b> $\text{Gr}(4, 8)$ $\text{Gr}(3, 9) \cong \text{Gr}(6, 9)$	$E_7^{(1,1)}$ $E_8^{(1,1)}$	<b>planar</b> square tiling triangular $\cong$ hexagonal tiling	$C_2^{(1)}$ $G_2^{(1)}$	$[4, 4]$ $[3, 6] \cong [6, 3]$
$r > 4$	<b>infinite mutation type</b>		<b>hyperbolic</b>		

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## References

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